Tutorial 8.1 Electric quadrupole moment

From Chapter 8 of *Physics of Electricity and Magnetism*, by John S. Colton (free at `https://em.byu.edu`).

Name: _____

Background

The quadrupole moment, (8.17), is a collection of nine components which can be represented by a matrix called a "rank 2 tensor" (i and j vary from 1 to 3, or equivalently from x to z). $\mathbf{r}'$ represents the locations where the charges are, r'_i and r'_j represent its Cartesian components (that is, x', y' or z'), and δ_{ij} is the Kronecker delta. If you have discrete charges instead of a continuous charge distribution, the integrals become summations over all the charges.

The potential produced by Q_{ij} at the location $\mathbf{r}$ is given by (8.18), where $\hat{\mathbf{r}}$ is a unit vector pointing toward the field point and (important!) $\hat{r}_i$ represents the ith Cartesian component of the unit vector. For example, if $\hat{\mathbf{r}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, then $\hat{r}_x = \frac{1}{\sqrt{2}}$, $\hat{r}_y = 0$, and $\hat{r}_z = \frac{1}{\sqrt{2}}$.

Part 1 — How the quadrupole moment tensor produces a field

To better understand what the different Q_{ij} tensor elements mean, first consider a field point on the x axis, $\mathbf{r}_1 = (x, 0, 0)$.

(a) Draw the quadrupole at the origin and the vector $\mathbf{r}_1$ to your field point, and identify the three Cartesian components of $\hat{\mathbf{r}}_1$. These should just be three numbers, which then become $\hat{r}_x$, $\hat{r}_y$ and $\hat{r}_z$ in the formula.

(b) Write out the formula for V_{quad} with all nine terms of the summation in (8.18) explicitly written out, and cross out the terms which are zero. What is the form of V_{quad} for this field point? Which component or components of Q_{ij} does it depend on?

(c) Similarly determine what the formula for V_{quad} becomes for these other field points: $\mathbf{r}_2 = (0, y, 0)$, $\mathbf{r}_3 = (0, 0, z)$, and $\mathbf{r}_4 = \frac{1}{\sqrt{2}}(r, r, 0)$.

(d) Describe in words what you have learned about when the tensor elements Q_{xx}, Q_{yy}, Q_{zz} and Q_{xy} affect the potential at a given point. Then complete the following sentence: *The tensor element Q_{ij} affects the potential at a given point* $\mathbf{r}$ *if* _____.

Part 2 — How a charge distribution produces the quadrupole moment tensor

Consider the following charge distribution, a sphere of radius R:

$$\rho(r, \theta) = \rho_0 \frac{r^2}{R^2} \left(3\cos^2\theta - 1\right).$$

(a) Make a sketch of where the charges are positive and negative within the sphere.

As hopefully seems plausible from the symmetry of your sketch, this charge distribution has no net charge, and hence no monopole moment q. This can also be shown algebraically:

$$\begin{aligned} q &= \int \rho(\mathbf{r}')\, d\tau' = \int \rho_0 \frac{r'^2}{R^2} \left(3\cos^2\theta' - 1\right) \left(r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'\right) \\ &= 0 \qquad \text{(from the } \theta' \text{ integral)}. \end{aligned}$$

(b) Make a physical argument from your sketch that the dipole moment should also be zero. One can show that that is actually the case by integrating $\mathbf{p} = \int \rho(\mathbf{r}')\,\mathbf{r}'\,d\tau'$, but you don't need to do that.

(c) Even though the monopole and dipole moments are zero for this charge configuration, the quadrupole moment will be nonzero. Why is that significant?

(d) Determine the nine components of the quadrupole moment tensor for this charge distribution, and write them in matrix form, $\overleftrightarrow{\mathbf{Q}}$. Mathematica (or any computer algebra system) is fair game for the integrals.

(e) What do you think would happen to $\overleftrightarrow{\mathbf{Q}}$ if the charge distribution were rotated to point in some arbitrary direction? It's fine if you don't have enough intuition yet to give a completely correct answer to this question; just give it your best guess.

Part 3 — Piecing together: V_{quad} of a specific charge distribution at a specific point

(a) Combine your results from Parts 1 and 2 to determine the quadrupole potential produced by the $\overleftrightarrow{\mathbf{Q}}$ of Part 2 at the specific point $\mathbf{r}_3 = (0, 0, z)$.

(b) Check whether your answer is reasonable, using at least one of the standard physics methods: limiting cases, units, comparison to other problems, and so on.