Tutorial 9.1 Polarization versus polarizability

From Chapter 9 of *Physics of Electricity and Magnetism*, by John S. Colton (free at `https://em.byu.edu`).

Name: _____

Background

If a dielectric material is exposed to an external electric field, the atoms will polarize. For linear, isotropic materials, the polarization field is given by $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$, where χ_e is the electric susceptibility and $\mathbf{E}$ is the electric field inside the material.

Meanwhile, if an individual atom is exposed to an electric field, the atom will polarize to create a dipole moment. For linear, isotropic situations, the dipole moment is given by $\mathbf{p} = \alpha \mathbf{E}$, where α is the atomic polarizability and $\mathbf{E}$ is the electric field at the location of the dipole.

There is a subtlety here, which is that the two electric fields in the equations for $\mathbf{P}$ and $\mathbf{p}$ are not the same. Focusing on one representative dipole now, the electric field to be used in the $\mathbf{P}$ equation is the overall spatially averaged electric field, which includes the field from that dipole itself, but the electric field to be used in the $\mathbf{p}$ equation is the field from *all the other sources*, including any applied external electric field and the field from the other dipoles, but *not* including its own field. Written out explicitly,

$$\mathbf{P} = \varepsilon_0 \chi_e \left(\mathbf{E}_{\text{self}} + \mathbf{E}_{\text{all other}}\right), \tag{1}$$

$$\mathbf{p} = \alpha \, \mathbf{E}_{\text{all other}}. \tag{2}$$

Meanwhile, because $\mathbf{P}$ is the density of dipole moments, $\mathbf{P}$ and $\mathbf{p}$ are related through

$$\mathbf{P} = N\mathbf{p}, \tag{3}$$

where N is the "number density" of dipole moments, namely the number of dipole moments per volume.

The goal of this tutorial is to use these concepts to figure out an equation that connects χ_e and α; it is called the **Clausius–Mossotti relation**.[a] This equation was an important development in the understanding of electric fields inside materials, since χ_e is a measurable property of bulk materials, and α is a property of isolated atoms or molecules, which can be measured in a dilute gas, or in a beam of atoms or molecules traveling through a vacuum.

[a] It is named for Ottaviano-Fabrizio Mossotti, in a memoir of 1846 (printed in 1850), and Rudolf Clausius, who arrived at it independently in a book of 1879. It is usually written in terms of ε_r instead of χ_e, using $\varepsilon_r = \chi_e + 1$ (Section 9.5.1). When it is written in terms of the index of refraction n, using $\varepsilon_r = n^2$ (which holds for a nonmagnetic material when ε_r is measured at the frequency of the light), it is also called the Lorentz–Lorenz equation, for Ludvig Lorenz and Hendrik Lorentz, who also found it independently of each other, in 1869 and 1878. All four have their names attached to it.

Part 1

We will start by supposing that the field from all the other sources is uniform near the atom and points in the z direction, which causes the atom to polarize, also in the z direction. We will model the polarized atom as a small spherical shell of radius R and say that the induced surface charge density of the shell is given by $\sigma = \sigma_0 \cos\theta$.

(a) Draw the sphere in the uniform electric field with the induced positive and negative charges on it. Does the form of σ seem reasonable? Note that in the context of the above equations, the uniform electric field is $\mathbf{E}_{\text{all other}}$ and the electric field from the spherical shell is $\mathbf{E}_{\text{self}}$.

(b) From the information given, consider how you might go about relating χ_e and α. Think about what equations, quantities, and relations you would want to use or find along the way. It is fine if your ideas here don't match the method presented below.

(c) Calculate the dipole moment of the spherical shell in terms of σ_0 and R. Use symmetry to simplify the integral, and do not violate Colton's First Commandment. You should be able to use this integral: $\int_0^\pi \cos^2\theta \sin\theta \, d\theta = 2/3$.

(d) Find the field at the center of the sphere from itself, in terms of σ_0. This is $\mathbf{E}_{\text{self}}$. There are a few ways of doing this, but please do it with the script-r method for finding $\mathbf{E}$ (Section 3.5). Normally I wouldn't recommend the script-r method, but since you only want the field at a single point, and it's a pretty easy point (the center), it turns out to be straightforward. You can again use symmetry to simplify the integral, and remember again not to violate Colton's First Commandment.

(e) Combine your answers from 1(c) and 1(d) to show that $\mathbf{E}_{\text{self}} = -\mathbf{p}/4\pi\varepsilon_0 R^3$.

(f) Describe with words what you have done and found out in this part of the problem.

Part 2

(a) The number density N is the number of atoms per volume; this is also the inverse of the volume per atom, $\frac{4}{3}\pi R^3$. Use this along with (2) and your answer to 1(e) to deduce the relationship between $\mathbf{E}_{\text{self}}$ and $\mathbf{E}_{\text{all other}}$ in terms of N and α: $\mathbf{E}_{\text{self}} = (\text{stuff})\,\mathbf{E}_{\text{all other}}$.

(b) Describe with words what this result means, and discuss whether it makes sense in terms of its sign and its dependences on N and α.

Part 3

(a) You are now in a position to use (1), (2), and (3), along with your result from Part 2, to determine the Clausius–Mossotti equation, which gives α as a function of χ_e. The equation will also depend on N and ε_0. *Hint*: Start with (1). Use the other equations to put everything in terms of $\mathbf{E}_{\text{all other}}$, which you can then cancel from all terms. What's left can be solved for α.

(b) Use at least one of the usual methods to consider whether the answer you obtained is reasonable.

Part 4

(a) How did the actual derivation differ from your original thoughts in 1(b) about how you might relate χ_e and α? What did you learn through this derivation that you didn't originally consider?

(b) Summarize the key differences between the electric fields used in the definitions of polarizability and susceptibility.