Tutorial 20.1 Forces from the Maxwell stress tensor

From Chapter 20 of *Physics of Electricity and Magnetism*, by John S. Colton (free at `https://em.byu.edu`).

Name: _____

Background

For static fields, the force on the charges and currents inside a Maxwellian surface is the integral of the stress tensor over the surface, (20.25), and the field that goes into the stress tensor is the total field on the surface. In this tutorial the total field is the sum of two fields: the field of the object whose force we want, and a field applied from outside. Each element of $\overleftrightarrow{\mathbf{T}}$ is quadratic in the fields, so splitting the field in two splits the stress tensor into three parts: an *applied part* $\overleftrightarrow{\mathbf{T}}^{\text{app}}$, made of the applied field alone; a *self part* $\overleftrightarrow{\mathbf{T}}^{\text{self}}$, made of the object's own field alone; and a *cross part* $\overleftrightarrow{\mathbf{T}}^{\text{cross}}$, each of whose terms contains one of each. You will find out which of the three parts push or pull on the object.

The force per unit area on a patch of the surface is the vector $\overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{n}}$, whose components are $\sum_j T_{ij} n_j$, with $\hat{\mathbf{n}}$ the outward unit normal (Section 20.3.3).

Part 1 — A point charge in a uniform field

A point charge $q > 0$ sits at the origin in the uniform field $\mathbf{E}_0 = E_0 \hat{\mathbf{z}}$, with $E_0 > 0$, produced by charges far away. For the Maxwellian surface, take a sphere of radius a centered on the charge.

(a) Sketch the field lines of the total field near the charge. Where is the field stronger than E_0, and where is it weaker? Is there a point where it is zero? From the picture of field lines as stretched elastic bands (Section 20.3.3), predict the direction of the force on the charge.

(b) Write the total field as $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_q$, where $\mathbf{E}_q$ is the field of the charge. Show that the stress tensor (20.19) splits into the three parts of the background, with the cross part

$$T_{ij}^{\text{cross}} = \varepsilon_0 \left(E_{0i} E_{qj} + E_{qi} E_{0j} - \delta_{ij} \mathbf{E}_0 \cdot \mathbf{E}_q \right).$$

(c) The applied part is the stress tensor of a uniform field, the same at every point. Explain why its integral over any closed surface is zero.

(d) On the sphere, $\mathbf{E}_q = (q/4\pi\varepsilon_0 a^2)\,\hat{\mathbf{n}}$. Find $\overleftrightarrow{\mathbf{T}}^{\text{self}} \cdot \hat{\mathbf{n}}$ there, and integrate it over the sphere. What does your result say about whether a charge can push or pull on itself?

(e) Find $\overleftrightarrow{\mathbf{T}}^{\text{cross}} \cdot \hat{\mathbf{n}}$ on the sphere. You should find that it is the same vector at every point of the sphere, even though $\hat{\mathbf{n}}$ points a different way at each point. Integrate it over the sphere to find the force on the charge. Does your result depend on a? Should it?

Part 2 — A wire in a uniform field

A long straight wire of radius R lies along the z axis and carries the current $I > 0$ in the $+z$ direction, in the uniform field $\mathbf{B}_0 = B_0\,\hat{\mathbf{x}}$, with $B_0 > 0$, produced by magnets far away. For the Maxwellian surface, take a cylinder of radius $a > R$ and length ℓ, centered on the wire and closed by two flat end caps.

(a) Sketch the field lines of the total field in the xy plane. On which side of the wire is the field stronger? Magnetic field lines pull along their length and push sideways on their neighbors, with the pressure $B^2/2\mu_0$. Predict the direction of the force on the wire.

(b) The stress tensor splits into applied, self, and cross parts again, with $1/\mu_0$ in place of ε_0 and the magnetic fields in place of the electric ones. Explain why the applied part gives no net force, and why the two end caps together give no net force from any of the three parts.

(c) On the curved surface, the wire's own field is $\mathbf{B}_I = (\mu_0 I/2\pi a)\,\hat{\boldsymbol{\phi}}$, and the outward normal is $\hat{\mathbf{n}} = \hat{\mathbf{s}}$, where

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \qquad \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}.$$

Show that the self part gives no net force.

(d) Find $\overleftrightarrow{\mathbf{T}}^{\text{cross}} \cdot \hat{\mathbf{n}}$ on the curved surface, and integrate it to find the force on the length ℓ of the wire. Does it depend on a? Write your result as a vector equation that does not depend on which way the x and y axes point.

Part 3 — Two parallel wires

Two long parallel wires, much thinner than the distance d between them, carry the currents I_1 and I_2 in the same direction. Wire 1 lies along the z axis, and wire 2 along the line $x = d$, $y = 0$. Surround a length ℓ of wire 1 with a cylinder of radius a, larger than wire 1 but smaller than d.

(a) Sketch the field lines of the total field in the xy plane. Is the field between the wires stronger or weaker than on their far sides? Predict whether the wires attract or repel.

(b) The total field is $\mathbf{B}_1 + \mathbf{B}_2$, and Part 2 showed that wire 1's self part gives no net force. The field $\mathbf{B}_2$ is not uniform across the cylinder, so the argument that you used for the applied part in Part 2 does not work here. Use the MST theorem, (20.21), to explain why the part made of $\mathbf{B}_2$ alone still gives no net force on the cylinder. (Apply the theorem to the field $\mathbf{B}_2$ alone, whose only current is in wire 2.)

(c) Explain why the force on wire 1 that you find from the cylinder cannot depend on a, as long as a stays between the radius of wire 1 and d.

(d) So make a small enough that $\mathbf{B}_2$ is nearly uniform across the cylinder. Use your result from 2(d) to find the force per unit length on wire 1, and its direction.

Part 4 — Wrap-up

(a) Compare your results with the forces you already know: (3.6) for a charge, (10.3) for a wire, and Example 10.2 for two parallel wires. Did your sketches predict the right directions? In all three parts only the cross part gave a force. In your own words, why?

(b) The negative of the stress tensor is the momentum flux density (Section 20.3.4). In Part 2, does momentum flow into the cylinder or out of it, and at what rate? If the wire is free to move, where does that momentum go, and where do you think it comes from?