Tutorial 23.1 Rectangular waveguides

From Chapter 23 of *Physics of Electricity and Magnetism*, by John S. Colton (free at `https://em.byu.edu`).

Name: _____

Background

Waveguides are hollow metal tubes that can be used to carry electromagnetic waves, typically in the microwave range of frequencies. A rectangular waveguide with inside dimensions a and b is shown in Figure 23.6. The waves are confined between four walls, at $x = 0$ and $x = a$ (the two vertical walls) and at $y = 0$ and $y = b$ (the two horizontal walls).

Just as any polarization of light can be decomposed into s- and p-polarizations, any field inside a waveguide can be decomposed into two kinds of modes, called TE and TM. In a TE mode ("transverse electric"), the electric field has no z component, while the magnetic field has x, y, and z components. In a TM mode ("transverse magnetic"), the magnetic field has no z component, while the electric field has x, y, and z components.

If the z components of the fields are known, the x and y components follow from the longitudinal-to-transverse equations (23.7). For TE modes the E_z terms in them are all zero, and for TM modes the B_z terms are all zero. So once we find B_z (for TE modes) or E_z (for TM modes), the complete solution for all components of both fields is easily obtained. For TE modes we find B_z from its Helmholtz equation (23.8), which comes from the wave equation $\nabla^2\mathbf{B} = (1/c^2)\partial^2\mathbf{B}/\partial t^2$, each component of which is a wave equation of its own, by assuming the plane-wave-like dependence $e^{i(kz-\omega t)}$ on z and t, with some other dependence on x and y.

Part 1 — General

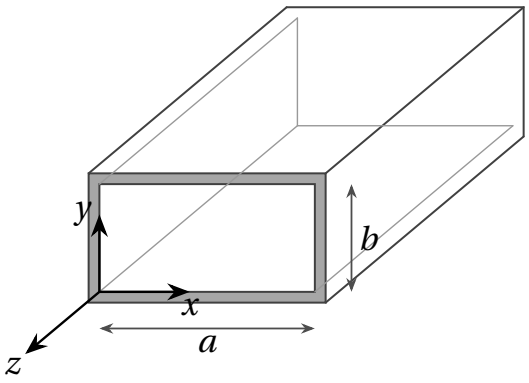


Figure 23.6 A rectangular waveguide, with inside width a and inside height b. The axes start at an inside corner of the guide, with z along it, toward the viewer.

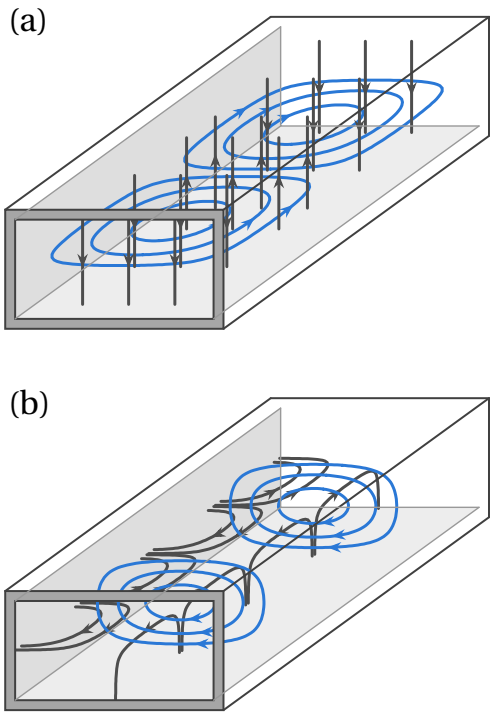


Figure 23.7 A representative TE mode and a representative TM mode of a rectangular waveguide at one instant, with the z axis along the tube. The black lines are electric field lines and the blue lines are magnetic field lines. The top and right walls are drawn transparent, to show the fields inside.

Figure 23.7 shows a representative TE mode and a representative TM mode, with the z axis along the tube. The black lines are electric field lines and the blue lines are magnetic field lines.

(a) Which picture is which mode? Explain your reasoning.

(b) Write down the four boundary conditions that come from Maxwell's equations. If you do not know any details about the free charge or free current at a surface, which two boundary conditions are most relevant?

(c) Describe how the two relevant boundary conditions are both satisfied by the TE and TM modes of Figure 23.7. Approximate the electric and magnetic fields inside the conductor as both being zero.

Part 2 — TE modes

(a) Now consider the TE modes specifically. As mentioned above, for the TE modes we focus on B_z. Assume that $B_z(x, y, z, t) = B_z(x, y)\, e^{i(kz-\omega t)}$, and show that the Helmholtz equation results from the wave equation. This notation is a little strange, because the B_z on the left is the complete field, while the B_z on the right is only its dependence on x and y across the guide, but unfortunately that is what is commonly done, including in this chapter.

(b) The Helmholtz equation is typically solved by separation of variables together with boundary conditions, so consider the boundary conditions further. For the TE modes we must consider the boundary condition on **B**. What constraint does it put on the components of **B** at the walls $x = 0$ and $x = a$? What constraint does it put on them at the walls $y = 0$ and $y = b$?

(c) The constraints you just found involve the x and y components of **B**, but the Helmholtz equation is for B_z. So use the longitudinal-to-transverse equations to connect the components to each other. Remembering that $E_z = 0$ for the TE modes, what do the longitudinal-to-transverse equations become?

(d) The TE modes of this guide have

$$B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),$$

where the integers m and n describe the details of the mode: how many nodes B_z has across the guide in the x and y directions. Each mode is labeled TE_{mn}, as in TE_{13} or TE_{25}. Show that this B_z, along with your previous results, makes the boundary conditions hold on the four walls. (It is enough to do $x = 0$ and $x = a$; the walls $y = 0$ and $y = b$ are similar.)

(e) This B_z looks as if it depends only on x and y, but as mentioned above, there is an underlying dependence on z and t as well. What is B_z when you include the z and t dependence explicitly? What do ω and k represent in your equation? It is not enough to say "angular frequency" and "wave number" here: what does the angular frequency mean in this context? What does the wave number mean?

Part 3 — Dispersion relation

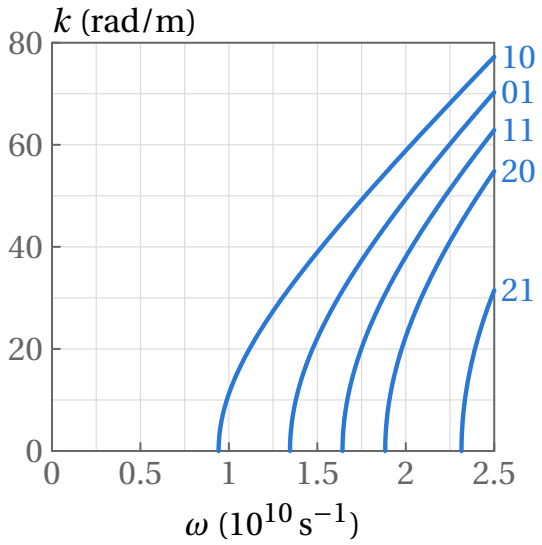


Figure 23.8 The wave number k against the angular frequency ω for the TE modes of a rectangular waveguide 10 cm wide and 7 cm high whose cutoff frequencies are below $2.5 \times 10^{10}\,\mathrm{s}^{-1}$. Each curve is labeled with its mode numbers mn.

Figure 23.8 shows k against ω for the TE modes of a waveguide 10 cm wide and 7 cm high. You will use it in step (c).

(a) Show that the full $B_z(x, y, z, t)$ of a TE mode does solve the Helmholtz equation, as long as k and ω are related by

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}, \qquad \text{where} \qquad \omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}.$$

(b) That equation for k as a function of ω is called the dispersion relation of the waveguide, and ω_{mn} is called the cutoff frequency of the mode. Note that ω_{mn} depends only on the mode numbers and on the size of the guide. What happens to k if ω is below the cutoff frequency of a mode? What does that mean for the physical behavior of the wave at that frequency?

(c) Look at the plot of k against ω in Figure 23.8. How many different TE modes can travel in that $10\,\text{cm} \times 7\,\text{cm}$ guide at the frequency $\omega = 1 \times 10^{10}\,\text{s}^{-1}$, and what are their wavelengths? How about at $\omega = 2 \times 10^{10}\,\text{s}^{-1}$?

Part 4 — Wrap-up

(a) Check that each wavelength you found in step 3(c) is longer than the wavelength $2\pi c/\omega$ of a wave of the same frequency in empty space. Why should it be?

(b) As ω increases far above every cutoff frequency, what happens to k for each mode? Does that make sense?