Tutorial 25.1 Relativistic particle decay

From Chapter 25 of *Physics of Electricity and Magnetism*, by John S. Colton (free at `https://em.byu.edu`).

Name: _____

Background

In a decay, one particle, called the parent, turns into two or more daughter particles; in a collision, two parents turn into daughters. The four-momentum of a group of particles is the sum of the four-momenta of the particles, and the center-of-momentum frame of the group is the one in which its total momentum is zero. Just as energy and momentum are conserved in introductory mechanics, four-momentum is conserved here: it is an experimental law, which holds in every frame of reference. The energy E, the momentum p, and the mass m of each particle are related by (25.31), $E^2 = p^2c^2 + m^2c^4$. This tutorial uses these facts to answer three questions:

1. Since mass and energy can be transformed into each other, could we make more massive decay products by giving an unstable particle more kinetic energy?
2. Why does a particle need to accelerate in order to emit photons?
3. Can we make more massive particles by colliding two highly energetic parent particles?

Part 1 — One parent, two daughters

A parent particle of mass M decays into two daughter particles with masses m_1 and m_2. The parent travels at speed v in the lab frame. In an attempt to answer question 1, we have given the parent a lot of kinetic energy: v is close to c in the lab frame.

(a) There are four reference frames that could be used to analyze this situation: the lab frame, the parent's frame, the frame of daughter 1, and the frame of daughter 2. Which is likely to be the simplest? Choose that frame. (If you have any doubts about whether you are making the best choice, please check.) Draw the situation in that frame, and write down the initial four-momentum, the four-momentum of the parent particle.

(b) For the sake of consistent notation, write the three-momenta of the daughter particles in your chosen frame as $\mathbf{p}_1 = p_{1x}\hat{\mathbf{x}} + p_{1y}\hat{\mathbf{y}} + p_{1z}\hat{\mathbf{z}}$ and $\mathbf{p}_2 = p_{2x}\hat{\mathbf{x}} + p_{2y}\hat{\mathbf{y}} + p_{2z}\hat{\mathbf{z}}$. Write down the four-momenta of the two daughters, using (25.31) to express their time components in terms of m_1, m_2, p_1, and p_2.

(c) Now apply conservation of four-momentum: set your answer to 1(a) equal to the sum of your answers to 1(b). The time component represents conservation of energy, and the three space components can be combined into one vector equation representing conservation of momentum.

(d) Prove that the conservation of energy equation has the consequence that $m_1 + m_2 \leq M$. To do this, think about what you need to do with p_1 and p_2 to make m_1 and m_2 as large as possible. This means that the total mass of the daughter particles can never be more than the parent's mass, even if the parent has a lot of kinetic energy in the lab frame. We have the answer to question 1: the decay of a single particle cannot be used to convert kinetic energy into mass.

(e) Does that statement generalize to more than two daughters? In other words, will it be true that $m_1 + m_2 + m_3 + \cdots \leq M$? Why or why not?

(f) The conservation of momentum equation also has an important physical consequence: in this frame the momenta are equal and opposite, $\mathbf{p}_1 = -\mathbf{p}_2$, whatever the daughters' masses, so that $p_1 = p_2$ in magnitude. Two daughters of different masses still have the same magnitude of momentum. Does that statement generalize to more than two daughters? In other words, if there are several daughters, will it be true that $p_1 = p_2 = p_3 = \cdots$ in this frame, whatever the values of m_1, m_2, m_3, and so on? Why or why not?

(g) Define $m_{\text{tot}} = m_1 + m_2$ and $\Delta m = m_1 - m_2$. Can Δm ever be larger than m_{tot}? Why or why not? Can Δm ever be larger than M? Why or why not?

(h) Rewrite your conservation of energy equation in terms of m_{tot} and Δm.

(i) Use Mathematica's `Solve` command, or `sympy.solve` in Python, to solve the conservation of energy equation for p_1 (which also equals p_2) in terms of M, m_{tot}, and Δm. You will likely get two possible answers. Are either or both of them physically reasonable? How do you know? Can the constraints on m_{tot} and Δm help you decide?

(j) Show that the physically reasonable solution can be put in the form

$$p_1 = p_2 = \frac{Mc}{2}\sqrt{\left[1-\left(\frac{m_{\text{tot}}}{M}\right)^2\right]\left[1-\left(\frac{\Delta m}{M}\right)^2\right]}. \tag{1}$$

One nice thing about this form is that everything inside the square root is dimensionless, so both sides of the equation explicitly have the dimensions of momentum.

Part 2 — Reflection questions

Use (1) to address the following questions.

(a) What happens to the magnitudes of the momenta, p_1 and p_2, as the daughters' total mass approaches M?

(b) What happens to p_1 and p_2 if both daughters are massless?

(c) Now consider a charged particle of mass M that emits a gamma ray (a photon). What are m_1 and m_2 in this scenario? What would the momentum of the photon need to be? Is that really a photon?

We now have the answer to question 2. Your equations should show that the decay of a particle cannot convert its kinetic energy into other particles, even massless ones, however much kinetic energy it has in the lab frame. So a charge cannot emit a photon and remain otherwise unchanged unless it is accelerating: as long as the charge's rest frame is an inertial frame, special relativity forbids it.

Part 3 — Two parents, two daughters

An important extension of this problem is to $2 \to 2$ scattering. In an attempt to answer question 3, let two incoming (parent) particles, with a lot of kinetic energy, collide and turn into two daughter particles. Let the parents' total energies be E_1 and E_2, and their momenta $\mathbf{P}_1$ and $\mathbf{P}_2$; the daughters' masses and momenta are m_1, m_2, $\mathbf{p}_1$, and $\mathbf{p}_2$, as in Part 1. Again analyze the situation in the center-of-momentum frame.

(a) What is the relationship between $\mathbf{P}_1$ and $\mathbf{P}_2$ in this frame? Define $E = E_1 + E_2$, and write down the total four-momentum of the incoming particles.

(b) Show that this four-momentum is mathematically equivalent to the four-momentum you found in 1(a), with the substitution $E = Mc^2$.

(c) Since the situation is mathematically equivalent to Part 1 with that substitution, the solution for p_1 and p_2 is (1) with M replaced by E/c^2:

$$p_1 = p_2 = \frac{E}{2c}\sqrt{\left[1 - \left(\frac{m_{\text{tot}}c^2}{E}\right)^2\right]\left[1 - \left(\frac{\Delta m\, c^2}{E}\right)^2\right]}.$$

What is the new constraint on m_{tot}? What about the constraint on Δm?

(d) We are finally ready to answer question 3. Based on your results, if you increased the kinetic energy of the incoming particles, would it be possible to make more massive outgoing particles?

Part 4 — Wrap-up

(a) Are there ways you can think of to check (1) and your answers in Part 2? (Limiting cases, units, special cases you can work out another way, and so forth.)