Tutorial 26.1 Tensor equations

From Chapter 26 of *Physics of Electricity and Magnetism*, by John S. Colton (free at `https://em.byu.edu`).

Name: _____

Background

Sections 26.3–26.6 wrote several of the most important equations of electromagnetism in tensor form. Each tensor equation stands for several ordinary equations, one for each value of its free index, and each of its terms hides a sum over every repeated index. In this tutorial you unpack five of them, as Section 26.4 did for $\partial_\mu J^\mu = 0$. The tools are the summation convention (Section 25.7.2); the four-gradient ∂_μ and ∂^μ, (26.13); the field tensor $F^{\mu\nu}$, (26.7), and the dual tensor $G^{\mu\nu}$, (26.9); the four-current $J^\mu = (c\rho, \mathbf{J})$, the four-potential $A^\mu = (V/c, \mathbf{A})$, and the four-velocity $\eta^\mu = \gamma_u(c, \mathbf{u})$. Remember that lowering an index changes the sign of each component that has a 0 in that index.

Part 1 — Five tensor equations

For each of the five equations in the steps below:

1. Identify the quantities in the equation, and say in words what it means. (For $\partial_\mu J^\mu = 0$ one would say, "The four-divergence of the four-current is zero.")
2. Determine the units of the terms on both sides of the equation, and make sure that they match.
3. Use the definitions of the quantities and the summation convention to write the equation out in full.
4. Determine which equation or equations of earlier chapters it corresponds to, and give their equation numbers.

(a) $\partial_\mu A^\mu = 0$. Here $\partial_\mu A^\mu$ is a Lorentz scalar, which is zero in every frame for the potentials used in this book's chapters on radiation.

(b) $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$. Both sides are four-vectors, so this is four equations, one for each value of μ. Write the four as two: first the equation for $\mu = 0$, then the three for $\mu = 1, 2, 3$ as one vector equation.

(c) $\partial_\nu G^{\mu\nu} = 0$. Again consider $\mu = 0$ first, then $\mu = 1, 2, 3$.

(d) $K^\mu = q\,\eta_\nu F^{\mu\nu}$. Section 26.5 worked out the $\mu = 0$ component. Which equation do the components $\mu = 1, 2, 3$ correspond to? (Use $K^{1,2,3} = \gamma_u \mathbf{F}$, from (25.38).)

(e) $\partial^\nu \partial_\nu A^\mu = -\mu_0 J^\mu$. By (26.14), $\partial^\nu \partial_\nu$ is the d'Alembertian $\Box^2$, a scalar operator, so acting on a four-vector it gives a four-vector. Consider $\mu = 0$ first, then $\mu = 1, 2, 3$.

Part 2 — Wrap-up

(a) Why would physicists choose to write the equations of electromagnetism in tensor notation like this? What are some advantages and disadvantages?

(b) The two equations of (26.16) contain all four of Maxwell's equations. Is the tensor form simpler than the four equations? In what sense, and what does it hide?